英

令和５年４月入学

東北大学大学院工学研究科
量子エネルギー工学専攻入学試験

試験問題冊子

数学Ａ　MATHEMATICS A

令和５年２月28日(火)
Tuesday, February 28, 2023　　10:00 ― 11:30

Notice

1. Do not open this examination booklet until instructed to do so.
2. An examination booklet, answer sheets, draft sheets are provided. Put your examinee number on each of the answer sheets and the draft sheets.
3. Answer all problems.
4. At the end of the examination, reconfirm your examinee number and the problem numbers on the answer sheets. Put your answer sheets in numerical order on your draft sheet, place them beside the test booklet, and wait for collection by an examiner. Do not leave your seat before instructed to do so by the examiner.

数 学 A　MATHEMATICS A

1. Consider a curve on the xy plane given by

$$x = a(t - \sin t),\ y = a(1 - \cos t) \qquad (0 \leq t \leq 2\pi),$$

where t is a parameter and a is a positive constant. Solve the following problems.

(1) Draw the curve on the xy plane.

(2) Evaluate the length l of the curve.

(3) Evaluate the area of the domain D enclosed by the curve and the x axis.

2. In the three-dimensional Cartesian coordinate system (x, y, z), the vector field $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = (x^3 + 3y + z)\,\boldsymbol{i} + (5x + y^3 + 3z)\,\boldsymbol{j} + (2x + y + z^3)\,\boldsymbol{k}\ ;$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are the fundamental vectors in the x, y, and z directions, respectively. In addition, the curved surface S is given by

$$S = \{\ (x, y, z)\ |\ \ x^2 + y^2 + z^2 = 4,\ z \geq 1\}\ .$$

Here, the three-dimensional polar coordinate system is expressed as (r, θ, ϕ). Solve the following problems.

(1) Obtain $\nabla \cdot \boldsymbol{A}$ and $\nabla \times \boldsymbol{A}$ in the three-dimensional Cartesian coordinate system.

(2) Draw the curved surface S. In addition, find the range of θ, corresponding to the curved surface S.

(3) Express the position vector of the curved surface S with $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, θ, and ϕ.

(4) The area of the curved surface S is given by

$$\int_0^{2\pi} d\phi \int_{\theta_{\min}}^{\theta_{\max}} \boxed{\ ①\ }\ d\theta\ ,$$

where $\theta_{\max}$ and $\theta_{\min}$ are the maximum and minimum values of θ obtained in problem (2), respectively. Write the formula of $\boxed{\ ①\ }$. In addition, obtain the area of the curved surface S.

(5) Evaluate the integral $\int_S \nabla \times \boldsymbol{A} \cdot \boldsymbol{n}\, dS$, where $\boldsymbol{n}$ is the unit normal vector of the curved surface S with a positive z component. In addition, evaluate the integral $\int_C \boldsymbol{A} \cdot d\boldsymbol{r}$, using the Stokes's theorem, where C is a loop around the curved surface S.

3. The 3×3 matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A}=\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} .$$

Solve the following problems.

(1) Find the eigenvalues of $\boldsymbol{A}$ by solving its characteristic equation.

(2) Show the following vectors are the eigenvectors of $\boldsymbol{A}$,

$$\boldsymbol{a}_1=\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} , \quad \boldsymbol{a}_2=\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} , \quad \boldsymbol{a}_3=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} .$$

(3) Find the orthogonal eigenvectors using the vectors given in problem (2).

(4) Find a 3×3 diagonal matrix $\boldsymbol{D}$ and a 3×3 matrix $\boldsymbol{P}$, which satisfy $\boldsymbol{A}=\boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$.

(5) Find $\boldsymbol{A}^n$.